Math 005C Exam 1

1. (4 pts) Find a vector of length 4 in the direction $\overrightarrow{PQ}$ for $P(0,1,2)$ and $Q(1,3,4)$.

$$\overrightarrow{PQ} = \langle 1, 2, 2 \rangle$$

$$|\overrightarrow{PQ}| = \sqrt{1+4+4} = 3$$

$\frac{1}{3}\langle 1,2,2 \rangle$ has length 1.

$\frac{4}{3}\langle 1,2,2 \rangle = \boxed{\langle \frac{4}{3}, \frac{8}{3}, \frac{8}{3} \rangle}$ has length 4

2. Let $\mathbf{a} = \langle 3, 5, 1 \rangle$ and $\mathbf{b} = \langle -1, 4, 3 \rangle$.

a. (2 pts) Find the scalar projection of **a** onto **b.**

$$\text{comp}_{\vec{b}}\, \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{-3+20+3}{\sqrt{1+16+9}} = \frac{20}{\sqrt{26}}$$

b. (4 pts) Find the vector projection of **a** onto **b.**

$$\text{proj}_{\vec{b}}\, \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b} = \frac{20}{26}\langle -1, 4, 3 \rangle$$

$$= \boxed{\left\langle -\frac{10}{13}, \frac{40}{13}, \frac{30}{13} \right\rangle}$$

3. (4 pts) Find parametric equations of the line that passes through the points $(8, -1, 2)$ and $(3, -5, 4)$.

$$\vec{v} = \langle 8-3, -1+5, 2-4 \rangle = \langle 5, 4, -2 \rangle$$

$$\boxed{\begin{aligned} x &= 8+5t \\ y &= -1+4t \\ z &= 2-2t \end{aligned}}$$

4. (8 pts) Find the equation of the plane through the point $P(1, -2, 3)$ containing the line $x = 4 + \underline{2}t$, $y = \underline{-}t$, $z = 1 + \underline{3}t$.

$\vec{v}_1 = \langle 2, -1, 3 \rangle$ is in the plane.

$(4, 0, 1)$ is on the line (and in the plane)

So $\vec{v}_2 = \langle 1-4, -2-0, 3-1 \rangle = \langle -3, -2, 2 \rangle$ is in the plane.

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ -3 & -2 & 2 \end{vmatrix} = \langle -2+6, -(4+9), -4-3 \rangle$$

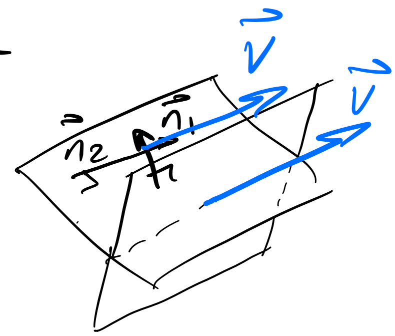
$$= 4\hat{i} - 13\hat{j} - 7\hat{k}$$

$$4(x-1) - 13(y+2) - 7(z-3) = 0$$

$$\boxed{4x - 13y - 7z = 9}$$

5. (8 pts) Find parametric equations for the line of intersection of the planes
$x + 2y + z = 4$ and $x - y + z = 1$.

the direction vector for the line is orthogonal to both normal vectors for the planes.



$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 3\hat{i} - 3\hat{k} \qquad (\text{use } \langle 1, 0, -1 \rangle)$$

Find a point on the line of intersection:

$$\begin{aligned} x + 2y + z &= 4 \\ x - y + z &= 1 \end{aligned}$$

Let $z = 0$: $\xrightarrow{\times(-1)}$

$$\begin{aligned} x + 2y &= 4 \\ -x + y &= -1 \\ \hline 3y &= 3 \\ y &= 1 \end{aligned}$$

$x - 1 = 1 \Rightarrow x = 2$

Check $x + 2 = 4 \Rightarrow x = 2$ So $(2, 1, 0)$ is on the line.

$$\boxed{\begin{aligned} x &= 2 + t \\ y &= 1 \\ z &= -t \end{aligned}}$$

Note: Different choices for a point on the line and a scaled direction vector yield different but equivalent representations.

6. Given $\mathbf{r}(t) = \langle t^2, \ln t, 2t \rangle$. Find the following.

Note that $t > 0$ so absolute value is not necessary for part (b).

a. (3 pts) The velocity of the particle in terms of t

$$\boxed{\vec{v}(t) = \vec{r}\,'(t) = \langle 2t, \tfrac{1}{t}, 2 \rangle}$$

b. (3 pts) The speed of the particle in terms of t

$$|\vec{v}(t)| = \sqrt{4t^2 + \frac{1}{t^2} + 4} = \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}}$$

$$= \frac{2t^2+1}{|t|} = \boxed{2t + \frac{1}{t}}$$

c. (4 pts) Parametric equations for the tangent line to $\mathbf{r}(t)$ at $t = 1$

$\vec{r}\,'(1) = \langle 2, 1, 2 \rangle$

$\vec{r}(1) = \langle 1, 0, 2 \rangle$ gives the point of tangency.

$$\boxed{\begin{aligned} x &= 1 + 2t \\ y &= t \\ z &= 2 + 2t \end{aligned}}$$

7. (6 pts) A particle moves with position function $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t^2\mathbf{k}$. Find a_T, the tangential component of the acceleration vector.

$$a_N = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{|r'(t)|}$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 2t \rangle$$

$$\vec{r}\,''(t) = \langle -\cos t, -\sin t, 2 \rangle$$

$$\vec{r}\,'(t) \cdot \vec{r}\,''(t) = \sin t \cancel{\cos t} - \cancel{\sin t}\cos t + 4t = 4t$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 4t^2} = \sqrt{1+4t^2}$$

$$\boxed{a_N = \frac{4t}{\sqrt{1+4t^2}}}$$

8. (8 pts) Find the curvature of $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k}$ at the point where $t = 0$.

$$K(t) = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$$

$$\vec{r}\,'(t) = \langle e^t, -e^{-t}, 1 \rangle \Rightarrow |\vec{r}\,'(t)| = \sqrt{e^{2t} + e^{-2t} + 1}$$

$$\text{so } |\vec{r}\,'(0)| = \sqrt{3}$$

$$\vec{r}\,''(t) = \langle e^t, e^{-t}, 0 \rangle$$

$$\vec{r}\,'(0) \times \vec{r}\,''(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \langle -1, 1, 2 \rangle$$

$$|\vec{r}\,'(0) \times \vec{r}\,''(0)| = \sqrt{6}$$

OR

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ e^t & -e^{-t} & 1 \\ e^t & e^{-t} & 0 \end{vmatrix} = \langle -e^{-t}, e^t, 2 \rangle$$

$$|\vec{r}\,'(t) \times \vec{r}\,''(t)| = \sqrt{e^{-2t} + e^{2t} + 4}$$

$$\Rightarrow |\vec{r}\,'(0) \times \vec{r}\,''(0)| = \sqrt{6}$$

at $t = 0$, $\boxed{K = \frac{\sqrt{6}}{3\sqrt{3}}}$ OR $K = \frac{\sqrt{2}}{3}$